Fachbereich Mathematik und Statistik
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Real Algebraic Geometry II

Exercise Sheet 2

Valuation independence and pseudo-convergence

Let Q be a field. If not further specified, any vector space we consider is a Q-vector space.

Exercise 6

(4 points)

Recall that the polynomial ring $\mathbb{R}[x]$ is a subring of the ring $\mathbb{R}[[x]]$ of formal power series with real coefficients. Consider both of these as $\mathbb{R}$-vector spaces, so $\mathbb{R}[[x]] = \mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}[x]$ is the subspace of $\mathbb{R}[[x]]$ of sequences $\mathbb{N} \longrightarrow \mathbb{R}$ with finite support.
Let v be the valuation on $\mathbb{R}[x]$ given in Exercise 2 (sheet 1).

a) Show that v extends into a valuation v_1 on $\mathbb{R}[[x]]$, i.e. that there exists an extension of v to a valuation v_1 on $\mathbb{R}[[x]]$ such that $v_1(p) = v(p)$ for all $p \in \mathbb{R}[x]$, such that the extension

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[[x]], v_1)$$

is immediate.

b) Show that v extends into a valuation v_2 on $\mathbb{R}[[x]]$ such that the extension

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[[x]], v_2)$$

is *not* immediate.

Exercise 7

(4 points)

Let (V_1, v_1) and (V_2, v_2) be valued vector spaces such that $S(V_1) = S(V_2)$. Let $\mathcal{B}_1 \subseteq V_1 \setminus \{0\}$ be a valuation basis for (V_1, v_1) and let $\mathcal{B}_2 \subseteq V_2 \setminus \{0\}$ be a valuation basis for (V_2, v_2). Suppose that there exists a valuation preserving bijection

$$\tilde{h}\colon \mathcal{B}_1 \longrightarrow \mathcal{B}_2.$$

Let $h\colon V_1 \longrightarrow V_2$ be the vector space isomorphism extending h. Show that h is valuation preserving.

Exercise 8

(4 points)

Recall that a *cardinal* is an ordinal λ which is in bijection with no $\alpha \in \lambda$.
Let $(\Gamma, \leq)$ be a non-empty, totally ordered set.

a) Let $\lambda = |\Gamma|$ be the unique cardinal in bijection with Γ and let $f\colon \lambda \longrightarrow \Gamma$ be a bijective function. Show that for all $\alpha \in \lambda$, there is a well-ordered subset $B_\alpha \subseteq \Gamma$ such that for any $\beta < \alpha$, there is an $a \in B_\alpha$ with $f(\beta) \leq a$.

b) Show that there exists a well-ordered cofinal subset $A \subseteq \Gamma$.

c) Let $\mathrm{cf}(\Gamma)$ be the least cardinal such that there exists a well-ordered and cofinal subset of Γ of cardinality $\mathrm{cf}(\Gamma)$. This cardinal is called the *cofinality* of $(\Gamma, \leq)$. Compute $\mathrm{cf}(\omega)$, $\mathrm{cf}(\omega+1)$ and $\mathrm{cf}(\omega+\omega)$.

Exercise 9

(4 points)

Let (V, v) be a valued vector space. Let

$$S = \{a_\rho : \rho < \lambda\} \subseteq V$$

be a pseudo-convergent set.

a) Show that an $x \in V$ is a pseudo-limit of S if and only if for all $\rho < \lambda$, we have $v(x - a_\rho) < v(x - a_{\rho+1})$.

b) Suppose that $v(V \setminus \{0\}) \subseteq \mathbb{N}$ and let $x \in V$ be a pseudo-limit of S. Show that x is the only pseudo-limit of S in V.

c) Let p be prime, $Q = \mathbb{F}_p$ and $(V, v) = (\bigsqcup_{\gamma \in \omega+1} \mathbb{F}_p, v_{\min})$. Find all pseudo-limits of $\{a_\rho : \rho < \omega\}$ where $a_\rho \colon \omega + 1 \longrightarrow \mathbb{F}_p$ is defined for all $\beta < \omega + 1$ by $a_\rho(\beta) = 1$ if $\beta = \rho$ and $a_\rho(\beta) = 0$ otherwise.

Exercise 10 (bonus)

(4 points)

Let $(\Gamma, \leq)$ be a linearly ordered set. We define $\mathrm{cof}(\Gamma)$ as the smallest order type of a well-ordered subset of $(\Gamma, \leq)$. We say that an ordinal α is *regular* if, when seeing α as the totally ordered set $(\alpha, <)$, we have $\mathrm{cof}(\alpha) = \alpha$.

a) Show that $\mathrm{cof}(\Gamma) = 0$ if and only if $\Gamma = \varnothing$ and that $\mathrm{cof}(\Gamma) = 1$ if and only if $(\Gamma, \leq)$ has a maximum.

b) Show that $\mathrm{cof}(\alpha)$ is regular.

c) Show that $\mathrm{cof}(\Gamma)$ is the *only* regular order type of a well-ordered and cofinal subset of Γ.

d) Show that $\mathrm{cof}(\Gamma) = \mathrm{cf}(\Gamma)$.

Please hand in your solutions by ***Thursday, 4 May 2023, 10:00*** *(postbox 14 in F4).*