Fachbereich Mathematik und Statistik
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Real Algebraic Geometry II

Exercise Sheet 3

Pseudo-completeness and valued groups

Let Q be a field. If not further specified, any vector space we consider is a Q-vector space.

Exercise 11

(4 points)

Let (V, v) be a valued vector space. Let $S = (a_\rho)_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo-limit $s \in V$. Let $q \in Q^\times$ and let $x \in V$.

a) Show that $q\,S := (q\,a_\rho)_{\rho<\lambda}$ is pseudo-Cauchy, with pseudo-limit $q\,s$.

b) Show that $x + S := (x + a_\rho)_{\rho<\lambda}$ is pseudo-Cauchy, with pseudo-limit $x + s$.

c) Suppose that 0 is a pseudo-limit of $x + q\,S := (x + q\,a_\rho)_{\rho<\lambda}$. Show that $\frac{-1}{q}\,x$ is a pseudo-limit of S.

d) Let $T = (b_\rho)_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo-limit $t \in V$. Is

$$S + T := (a_\rho + b_\rho)_{\rho<\lambda}$$

necessarily pseudo-Cauchy with pseudo-limit $s + t$? Justify your answer.

Exercise 12

(4 points)

Let $p \in \mathbb{N}$ be a prime and let $(G, +, 0)$ be an abelian p-group, i.e. an abelian group such that for all $g \in G$, there exists an $n \in \mathbb{N}$ with $p^n\,g = 0$. Suppose that

$$\bigcap_{n\in\mathbb{N}} p^n\,G = \{0\}.$$

Let $P(G)$ be the subgroup of elements of G with $p\,g = 0$. This is a vector space over $\mathbb{F}_p$ in a natural way. The height function h on G is defined by $h(g) := n$ if $g \in p^n\,G \setminus p^{n+1}\,G$ for some $n \in \mathbb{N}$, and $h(g) = \infty$ otherwise. Show that $(P(G), h)$ is a valued $\mathbb{F}_p$-vector space.

We will follow up on this exercise in subsequent sheets, in order to prove Ulm' s theorem on the classification of countable abelian reduced p groups.

Please hand in your solutions by ***Thursday, 11 May 2023, 10:00*** *(postbox 14 in F4).*