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Real Algebraic Geometry II

Exercise Sheet 4

Abelian groups

Exercise 13

(5 points)

The purpose of this exercise is to prove a part of Ulm's theorem. It is a follow-up to Exercise 12, sheet 3.

Let $p \in \mathbb{N}$ be a prime number, let G be an abelian p-group, and denote by $P(G)$ the socle of p, i.e. the subgroup of G of elements of order p, i.e. the p-torsion subgroup.

For each ordinal α, define $p^{\alpha} G$ as follows

- $p^0 G := G$,
- if $\alpha = \beta + 1$ is a successor ordinal, then $p^{\alpha} G := p\,(p^{\beta} G)$,
- if α is a non-zero limit, then $p^{\alpha} G := \bigcap_{\beta < \alpha} p^{\beta} G$.

a) Show that there is an ordinal α such that $p^{\alpha} G = p^{\beta} G$ for all ordinals $\beta \geqslant \alpha$. The least such ordinal is called the *length* of G and denoted $\ell(G)$. We say that G is *reduced* if $p^{\ell(G)} G = \{0\}$.

   For every $\alpha < \ell(G)$, write $P_{\alpha}(G) := P(G) \cap p^{\alpha} G$ and define the **α-th homogeneous component** of G as the quotient space

$$P_{\alpha}(G) / P_{\alpha+1}(G),$$

   and define the **α-th Ulm invariant** as its (possibly infinite) dimension over $\mathbb{F}_p$.

b) Assume that G is reduced. Let $h_G: P(G) \longrightarrow \ell(G) \cup \{\infty\}$ be the function defined by $h_G(g) = \alpha$ if $g \in p^{\alpha} G \setminus p^{\alpha+1} G$, and $h(0) = \infty$. As in Exercise 12, Sheet 3, show that h_G is well-defined and that $(P(G), h_G)$ is a valued $\mathbb{F}_p$-vector space.

c) Show that if G is countable and reduced, then $(P(G), h_G)$ is completely determined up to isomorphism of $\mathbb{F}_p$-valued spaces by $\ell(G)$ and the family of Ulm invariants.

d) *It can be shown that in turn, a countable reduved p-group G is completely determined up to isomorphism by $(P(G), h_G)$ as a valued $\mathbb{F}_p$-vector space (but this is not true in general).*

Exercise 14

(5 points)

Let G be an ordered abelian group. Let $C \subseteq G$ be a convex subgroup and $B = G / C$.

a) Define the relation $\leqslant_B$ on B setting

$$g_1 + C \leqslant_B g_2 + C \qquad \text{whenever} \qquad g_1 \leqslant g_2.$$

Show that $(B, \leqslant_B)$ is a totally ordered abelian group.

b) Show that the set of convex subgroups of G is totally ordered by the relation $\subseteq$.

c) Find a bijective correspondence between convex subgroups of B and convex subgroups C' of G with $C \subseteq C'$.

d) Let D_1 and D_2 be convex subgroups of G such that $D_1 \subseteq D_2$ and that there is no further subgroup of G between D_1 and D_2. Show that D_2 / D_1 has no non-trivial convex subgroup.

e) Show that G is Archimedean if and only if its only convex subgroups are $\{0\}$ and G.

Exercise 15

(3 points)

Let G be an ordered abelian group, and let $x \in G \setminus \{0\}$.

a) Show that C_x and D_x are convex subgroups of G with $D_x \subsetneq C_x$.

b) Show that D_x is the largest proper convex subgroup of C_x (for the inclusion).

c) Deduce that the ordered abelian group C_x / D_x is Archimedean.

Exercise 16

(4 points)

Let G be an ordered abelian group.

a) Let v be as defined in Script 9, Proposition 3.5. Show that v is a valuation on G, i.e. that (G, v) is a valued $\mathbb{Z}$-module.

b) Let $x \in G \setminus \{0\}$. Show that

$$G^{v(x)} = \bigcap \{C : C \text{ is a convex subgroup of } G \text{ and } x \in C\}$$

and

$$G_{v(x)} = \bigcup \{C : C \text{ is a convex subgroup of } G \text{ and } x \notin C\}.$$

Conclude that $B_x = B(G, v(x))$ and that B_x is Archimedean.

Exercise 17 (bonus)

(4 points)

Let (V, v) and (W, w) be valued vector spaces, and assume that (V, v) is maximally valued and isomorphic to (W, w). Show directly (without using Hahn's embedding theorem) that (W, w) is maximally valued.

*Please hand in your solutions by **Friday, 19 May 2023, 10:00** (postbox 14 in F4).*