Fachbereich Mathematik und Statistik
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SoSe 2023

Real Algebraic Geometry II

Exercise Sheet 5

Ordered abelian groups and valued fields

Exercise 17
(4 points)

Let $[\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$ be an ordered system of Archimedean ordered abelian groups. Let

$$G := \coprod_{\gamma \in \Gamma} B(\gamma)$$

and define a relation $<_{\text{lex}}$ on G by

$$0 <_{\text{lex}} g :\iff (g \neq 0 \wedge 0 < g(v_{\min}(g))).$$

a) Show that $(G, <_{\text{lex}})$ is an ordered abelian group.

b) Show that $v_{\min}$ and the natural valuation v on G are equivalent.

Exercise 18
(4 points)

Let $p \in \mathbb{N}$ be prime. Define the map v_p on $\mathbb{Q}$ as follows

- Let $v_p(0) = +\infty$.
- For any $k \in \mathbb{Z} \setminus \{0\}$, let $v_p(k) = \max\{\ell \in \mathbb{N}_0 : p^l \text{ divides } k \text{ in } \mathbb{Z}\}$.
- For any $k, m \in \mathbb{Z} \setminus \{0\}$, let $v_p(\frac{k}{m}) = v_p(k) - v_p(m)$.

a) Justify that v_p is well-defined.

b) Show that v_p is a valuation on $\mathbb{Q}$.

c) Determine R_{v_p}, I_{v_p} and K_{v_p}.

Exercise 19
(4 points)

Let $K = \mathbb{R}((X))$. So the underlying real vector space of K is the Hahn product $K = \mathbf{H}_{n\in\mathbb{Z}} \mathbb{R}$.

a) Let $v_{\min}$ be the map on K defined in Real Algebraic Geometry I, Lecture 24, Notation 1.4. Show that $(K, v_{\min})$ is a valued field and determine its value group $G_{v_{\min}}$ and its residue field $K_{v_{\min}}$.

b) Consider K as an ordered field with the ordering induced by $0 < X < r$ for all $r \in \mathbb{R}$ with $r > 0$. Let v be the natural valuation on K. Determine the value group G_v and the residue field K_v.

c) Show that

$$\varphi\colon\ G_{v_{\min}} \longrightarrow G_v,\ \ v_{\min}(x) \mapsto v(x)$$

is an order-preserving isomorphism of groups and that

$$\psi\colon\ K_{v_{\min}} \longrightarrow K_v,\ \ a v_{\min} \mapsto a v$$

is an order-preserving isomorphism of fields.

Exercise 20
(5 points)

Show that there is no non-trivial valuation on any algebraic extension $K\,/\,F$ of a finite field F.

*Please hand in your solutions by **Thursday, 25 May 2023, 10:00** (postbox 14 in F4).*