Fachbereich Mathematik und Statistik
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Real Algebraic Geometry II

Exercise Sheet 6

Hardy fields and Neumann's lemma

Exercise 21

(4 points)

Let H be a Hardy field.

a) Recall the definition of asymptotic equivalence $\sim$ on H (Real Algebraic Geometry II, Script 14, Definition 2.1). Show that $\sim$ coincides with the Archimedean equivalence relation on H.

b) Hence show that $(v(H^\times), +, <)$ is an ordered abelian group and that v is a valuation on H.

c) Show that

$$\begin{aligned} R_v &= \{f \in H : \lim_{x \to +\infty} f(x) \in \mathbb{R}\}, \\ I_v &= \{f \in H : \lim_{x \to +\infty} f(x) = 0\}, \quad \text{and} \\ \mathcal{U}_v &= \{f \in H : \lim_{x \to +\infty} f(x) \in \mathbb{R}^\times\}. \end{aligned}$$

Exercise 22

(4 points)

Let G be an ordered abelian group. Let $A, B \subseteq G$ be well-ordered subsets. Show that

$$A + B := \{a + b : (a, b) \in A \times B\}$$

is a well-ordered subset of G.

Exercise 23

(4 points)

Let k be an Archimedean ordered field and let G be a non-trivial ordered abelian group.

a) Show that $<_{\text{lex}}$ is an ordered field ordering on $k((G))$, i.e. that for $a, b, c \in k((G))$, we have

- if $a <_{\text{lex}} b$, then $a + c <_{\text{lex}} b + c$
- if $0 <_{\text{lex}} a$ and $0 <_{\text{lex}} b$, then $0 <_{\text{lex}} a\, b$.

b) Let $\varepsilon \in k((G))$ with $\operatorname{supp} \varepsilon \subseteq G^{>0}$. Show that

$$(1 - \varepsilon) \left(\sum_{n=0}^{+\infty} \varepsilon^n\right) = 1.$$

c) Let $g_1, g_2 \in G$. Compute $(t^{g_1} + t^{g_2})^{-1}$.

Exercise 24

(4 points)

Let G be a non-trivial ordered abelian group, and let $K = \mathbb{R}((G))$. For any $\varepsilon \in I_v$, define

$$e(\varepsilon) := \sum_{n=0}^{+\infty} \frac{\varepsilon^n}{n!}$$

a) Show that e is a well-defined function from I_v to $1 + I_v$.

b) Show that e is an order-preserving homomorphism from $(I_v, +, 0, <)$ to $(1 + I_v, \cdot, 1, <)$.

c) *Bonus question:* Show that

$$\ell\colon 1 + I_v \longrightarrow I_v\ ;\ 1 + \varepsilon \mapsto \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \varepsilon^n$$

is the inverse function of e (which is thus bijective).

Please hand in your solutions by ***Thursday, 01 June 2023, 10:00*** *(postbox 14 in F4).*