Fachbereich Mathematik und Statistik
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Real Algebraic Geometry II

Exercise Sheet 9

Convex valuations

Exercise 28

(4 points)

Let K be a field with valuations w_1 and w_2.

a) Show that the following are equivalent:

   i. w_2 is coarser than w_1,

   ii. $I_{w_2} \subseteq I_{w_1}$,

   iii. for all $a, b \in K$, we have

$$w_1(a) \leqslant w_1(b) \Longrightarrow w_2(a) \leqslant w_2(b).$$

b) Suppose that w_2 is coarser than w_1. Let

$$\varphi: K_{w_2} \longrightarrow K w_2 ; a \mapsto a w_2$$

be the residue map for w_2, where K_{w_2} denotes the valuation ring and $K w_2$ the residue field of (K, w_2). Show that $\varphi(K_{w_1})$ is a valuation ring of $K w_2$.

Exercise 29

(4 points)

a) Let $\mathbb{K} = \mathbb{R}((\mathbb{Q} \times \mathbb{R}))$, where $\mathbb{Q} \times \mathbb{R}$ is ordered lexicographically. Let

$$C = \{(0, r) : r \in \mathbb{R}\}.$$

   i. Compute the convex valuation w on $\mathbb{K}$ associated to C.

   ii. Find the value group and the residue field of $(\mathbb{K}, w)$.

   iii. Compute the rank of $(\mathbb{K}, w)$.

b) Let $K = \mathbb{R}(t)$. Show that for any ordering on K, the rank of K is the singleton $\{K\}$.

Exercise 30

(4 points)

Let $A = \mathbb{R}[x, x^{-1}, \mathrm{e}^x, \log x]$ be the ring of germs at $+\infty$ generated by the germs x of the identity, x^{-1} of the inverse function, e^x of the exponential function and $\log x$ of the natural logarithm.

a) Let $k \in \mathbb{Z}$ and $l, m \in \mathbb{N}$. Show that the germ $x^k \, \mathrm{e}^{l x} (\log x)^m$ is infinite if and only if $(l, k, m) >_{\mathrm{lex}} (0, 0, 0)$ in the lexicographic ordering $<_{\mathrm{lex}}$ on $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$.

b) Show that A is closed under derivation.

c) Show that each $f \in A$ which is non-zero is invertible as a germ.

d) Show that the fraction field H of A in the ring of germs is a Hardy field.

e) Compute the rank of H with its natural valuation.

Exercise 31 (bonus)
(6 points)

Let (K, v) be a valued field with valuation ring R_v and write $k = Kv$ for its residue field. Let w be a valuation on k with valuation ring R_w.

a) Let $\varphi: R_v \longrightarrow k$ be the residue map and write $R := \varphi^{-1}(R_w) \subseteq R_v$. Show that R is a valuation ring on K. We write $v \circ w$ for the corresponding valuation, called the *compositum* of v with w.

b) Show that $v \circ w$ is finer than v and determine the corresponding value group.

c) Assume that $k = \mathbb{R}((\mathbb{R}))$ and $K = k((\mathbb{Q}))$, and that w and v are the respective $v_{\min}$ valuations. What is the valuation $v \circ w$ on K?

Please hand in your solutions by ***Thursday, 29 June 2023, 10:00*** *(postbox 14 in F4).*