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Real Algebraic Geometry I

Exercise Sheet 1
Orderings and Dedekind cuts

Exercise 1
(4 points)
Let $n \in \mathbb{N}$ and let $\mathrm{sym}_{n\times n}(\mathbb{R})$ denote the set of all symmetric $(n \times n)$-matrices over $\mathbb{R}$. A matrix $A \in \mathrm{sym}_{n\times n}(\mathbb{R})$ is called **positive semi-definite (psd)** if $v^t A v \geq 0$ for any $v \in \mathbb{R}^n$. Consider the relation $\leq$ on $\mathrm{sym}_{n\times n}(\mathbb{R})$ given by

$$A \leq B :\Longleftrightarrow B - A \text{ is psd.}$$

(a) Show that for every $n \geq 2$, the relation $\leq$ is a partial order on $\mathrm{sym}_{n\times n}(\mathbb{R})$ which is NOT total.

(b) Determine for what $A \in \mathrm{sym}_{n\times n}(\mathbb{R})$ the set $\{\lambda A \mid \lambda \in \mathbb{R}\}$ is totally ordered by $\leq$.

Exercise 2
(4 points)

(a) Let $\preceq$ be the relation on $\mathbb{R}[\mathrm{x}]$ defined as follows: For any $p(\mathrm{x}) = a_n\mathrm{x}^n + \ldots + a_1\mathrm{x} + a_0$ and $q(\mathrm{x}) = b_m\mathrm{x}^m + \ldots + b_1\mathrm{x} + b_0$,

$$p(\mathrm{x}) \preceq q(\mathrm{x}) :\Longleftrightarrow a_i \leq b_i \text{ for all } i \in \mathbb{N}_0,$$

where we set $a_i = 0$ for $i > n$ and $b_i = 0$ for $i > m$. Show that $\preceq$ defines a partial order on $\mathbb{R}[\mathrm{x}]$ which is NOT total.

(b) We denote the set of **formal power series in one variable** by

$$\mathbb{R}[\![\mathrm{x}]\!] := \left\{ \sum_{i=0}^{\infty} a_i \mathrm{x}^i \;\middle|\; a_i \in \mathbb{R} \right\}.$$

Note that $\mathbb{R}[\mathrm{x}]$ is a subset of $\mathbb{R}[\![\mathrm{x}]\!]$.

Let $\leq$ be the total order on $\mathbb{R}[\mathrm{x}]$ given in Lecture 1, Example 3.6. Show that $\leq$ can be extended to a total order on $\mathbb{R}[\![\mathrm{x}]\!]$, i.e. find a total order $\leq'$ on $\mathbb{R}[\![\mathrm{x}]\!]$ such that for any $p(\mathrm{x}), q(\mathrm{x}) \in \mathbb{R}[\mathrm{x}]$,

$$p(\mathrm{x}) \leq' q(\mathrm{x}) \Longleftrightarrow p(\mathrm{x}) \leq q(\mathrm{x}).$$

(c) Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real-valued sequences, (i.e., functions $s\colon \mathbb{N} \to \mathbb{R}$). Find a total order $\leq$ on $\mathbb{R}^{\mathbb{N}}$ and an order-preserving bijection

$$\varphi : \left(\mathbb{R}^{\mathbb{N}}, \leq\right) \to (\mathbb{R}[\![\mathrm{x}]\!], \leq'),$$

i.e. a bijection φ from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}[\![\mathrm{x}]\!]$ such that for any $a, b \in \mathbb{R}^{\mathbb{N}}$, if $a \leq b$, then $\varphi(a) \leq' \varphi(b)$, where $\leq'$ is the total order from (b).

(d) For a sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we define the **support of** a by

$$\mathrm{supp}(a) := \{n \in \mathbb{N} \mid a_n \neq 0\}.$$

Let $F \subseteq \mathbb{R}^{\mathbb{N}}$ be the set of all sequences with finite support. Describe the totally ordered set $(\varphi(F), \leq')$.

Exercise 3
(4 points)
Let $(K, \leq)$ be an ordered field and denote by $K^{\times}$ the multiplicative group $K \setminus \{0\}$. Prove that the following are equivalent:

(i) $(K, \leq)$ is Archimedean.

(ii) For any $a, b \in K^{\times}$ there exists $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$.

(iii) K contains no infinitesimal positive element.

(iv) $\mathbb{Z}$ is coterminal in K.

Exercise 4
(4 points)

(a) Let $(\Gamma, \leq)$ be a totally ordered set. Show that $(\Gamma, \leq)$ is Dedekind complete if and only if $(\Gamma, \leq)$ has no free Dedekind cut.

(b) Let $(K, \leq)$ be a Dedekind complete ordered field. Show that $(K, \leq)$ is Archimedean.

*Please hand in your solutions by **Thursday, 03 November 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*