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Real Algebraic Geometry I

Exercise Sheet 2
Hölder's theorem and positive cones

Exercise 5
(4 points)
The aim of this exercise is to complete the proof of Hölder's theorem in detail.

(a) Let $(K, \leq)$ be an Archimedean ordered field. Show that $\mathbb{Q}$ is dense in $(K, \leq)$, i.e. for any $x, y \in K$ with $x < y$ there exists $q \in \mathbb{Q}$ such that $x < q < y$.

(b) Let $(K, \leq)$ be an Archimedean ordered field and let $\varphi : K \to \mathbb{R}$ be the map defined in the proof of Hölder's theorem, i.e. for any $a \in K$, we define $\varphi(a) := \sup I_a = \inf F_a \in \mathbb{R}$, where

$$I_a := \{q \in \mathbb{Q} \mid q \leq a\} \text{ and } F_a := \{q \in \mathbb{Q} \mid a \leq q\}.$$

Show that:

(i) φ is a ring homomorphism between K and $\mathbb{R}$ and therefore a field embedding.

(ii) φ preserves the order, i.e. for any $a, b \in K$, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.

Exercise 6
(4 points)
The aim of this exercise is to prove that $(\mathbb{R}, \leq)$ is the unique Dedekind complete ordered field up to isomorphism.

(a) Recall that $(\mathbb{R}, \leq)$ is supremum (least upper bound) complete, i.e. any nonempty subset of $\mathbb{R}$ which is bounded from above has a supremum (least upper bound) in $\mathbb{R}$. Deduce that $(\mathbb{R}, \leq)$ is Dedekind complete.

(b) Let $(K, \leq)$ be a Dedekind complete ordered field. Show that K is isomorphic to $\mathbb{R}$ as an ordered field, i.e. that there exists an order-preserving isomorphism from K to $\mathbb{R}$.

(Hint: Recall Exercise 4)

Exercise 7
(4 points)

(a) Show that a cone P in a field K is proper if and only if $-P \cap P = \{0\}$.

(b) Prove the following:

  (i) If $(K, \leq)$ is an ordered field, then the subset $P := \{a \in K \mid a \geq 0\}$ is a positive cone of K.

  (ii) If P is a positive cone of a field K, then the relation

$$a \leq b :\Longleftrightarrow b - a \in P$$

  defines an order on K such that $(K, \leq)$ is an ordered field.

(c) Deduce that, for any field K, there is a bijective correspondence between the set of orderings on K and the set of positive cones of K.

For a field K with a positive cone P we now also call (K, P) an ordered field, where the order on K induced by P is as above. In this case, we also say that P is an **ordering on** K.

Exercise 8
(4 points)
Let K be a field. Recall that the **set of sums of squares** of elements of a field K is denoted by $\sum K^2$. Show that:

(a) $\sum K^2$ is the smallest cone of K.

(b) If K is (formally) real, then $\sum K^2$ is a proper cone.

(c) If K is algebraically closed, then K is not real.

(d) If (K, P) is an ordered field, F is another field and $\varphi : F \to K$ is a field homomorphism, then $Q := \varphi^{-1}(P)$ is an ordering of F.

  In this case, we say that P is an **extension** of Q and Q is the **pullback** of P.

(e) If P_1 and P_2 are positive cones of K with $P_1 \subseteq P_2$, then $P_1 = P_2$. Deduce that if $\sum K^2$ is a positive cone, then it is the only ordering of K.

(f) The fields $\mathbb{R}$ and $\mathbb{Q}$ admit a unique ordering.

*Please hand in your solutions by **Thursday, 10 November 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*