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Real Algebraic Geometry I

Exercise Sheet 3
Extensions of orderings and Laurent series

Exercise 9
(4 points)
Let K be a field and let $\mathcal{T} = \{T_i \mid i \in I\}$ be a family of preorderings on K. Show that:

(a) The intersection $\bigcap_{i\in I} T_i$ is a preordering on K.

(b) If for any $i, j \in I$ there exists $k \in I$ such that $T_i \cup T_j \subseteq T_k$, then $\bigcup_{i\in I} T_i$ is a preordering of K.

Exercise 10
(4 points)
Show by induction on $n \in \mathbb{N}$: Any ordering on a field K extends to an ordering on the field of rational functions in several variables $K(\mathrm{x}_1, \ldots, \mathrm{x}_n)$.

Exercise 11
(4 points)
We proved in lecture 2 that each Dedekind cut of $\mathbb{R}$ corresponds to an ordering on $\mathbb{R}[\mathrm{x}]$ and in particular on $\mathbb{R}(\mathrm{x})$. Describe explicitly the ordering on $\mathbb{R}[\mathrm{x}]$ corresponding to each Dedekind cut of $\mathbb{R}$. Proceed as follows:

(a) Retrieve the orderings on $\mathbb{R}[\mathrm{x}]$ corresponding to 0_+ and 0_-, using the derivatives of a generic polynomial $p \in \mathbb{R}[\mathrm{x}]$ at 0.

(b) Using the same techniques as in (a), describe the orderings on $\mathbb{R}[\mathrm{x}]$ corresponding to all the remaining Dedekind cuts of $\mathbb{R}$.

(c) Conclude that, given an ordering on $\mathbb{R}[x]$ there exists a function

$$\sigma : \mathbb{R}[\mathrm{x}] \to \mathbb{R}$$

such that $\mathrm{sign}(p(\mathrm{x})) = \mathrm{sign}(\sigma(p))$ for any $p \in \mathbb{R}[\mathrm{x}]$.

Exercise 12
(4 points)
We denote the set of **real formal Laurent series** by

$$\mathbb{R}(\!(X)\!) := \left\{ \sum_{i=m}^{\infty} a_i X^i \;\middle|\; m \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$

For any $0 \neq A \in \mathbb{R}(\!(X)\!)$, we define $v(A)$ to be the smallest integer m such that $a_m \neq 0$. Moreover, for any

$$A = \sum_{i=m}^{\infty} a_i X^i \in \mathbb{R}(\!(X)\!) \text{ and } B = \sum_{i=n}^{\infty} b_i X^i \in \mathbb{R}(\!(X)\!)\,,$$

we define:

- the **coefficientwise addition**

$$A + B := \sum_{i=k}^{\infty} (a_i + b_i) X^i,$$

  where $k = \min\{m, n\}$ and we set $a_i = 0$ for $i < m$ and $b_i = 0$ for $i < n$;

- the **convolution product**

$$AB := \sum_{i=m+n}^{\infty} \left(\sum_{j+k=i} a_j b_k \right) X^i;$$

- the **order relation**

$$A \geq 0 :\Longleftrightarrow A = 0 \vee \left(A \neq 0 \wedge a_{v(A)} > 0 \right).$$

It can be shown that $\mathbb{R}(\!(X)\!)$ endowed with these operations and order relation is an ordered field and that $\mathbb{R}[\![X]\!]$ is a subring of $\mathbb{R}(\!(X)\!)$.

(a) Show that the map $v : \mathbb{R}(\!(X)\!)^{\times} \to \mathbb{Z}$ is a **discrete valuation** on $\mathbb{R}(\!(X)\!)^{\times} = \mathbb{R}(\!(X)\!) \setminus \{0\}$, i.e. that for any $A, B \in \mathbb{R}(\!(X)\!)^{\times}$, the following hold:

   (i) $v(A + B) \geq \min\{v(A), v(B)\}$.

   (ii) $v(AB) = v(A) + v(B)$.

(b) Deduce that $\mathbb{R}(\!(X)\!)$ is not real closed.

*Please hand in your solutions by **Thursday, 17 November 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*