Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 4
Real closed fields

Exercise 13
(4 points)
Let $(K, \leq)$ be an ordered field and let $\mathcal{B} := \{]a, b[\mid a, b \in K, a < b\} \cup \{\emptyset\}$, i.e. the collection of all open intervals and the empty set.

(a) Show that $\mathcal{B}$ forms the base of a topology on K, i.e. that

 (i) $\mathcal{B}$ is closed under finite intersections

 (ii) $\mathcal{B}$ covers K: $K \subseteq \bigcup_{A \in \mathcal{B}} A$.

 The topology on K induced by $\mathcal{B}$ is called the **order topology**. For the remainder of this exercise, we will consider K as a topological space, endowed with the order topology.

(b) Show that the field operations $+ : K \times K \to K$ and $\cdot : K \times K \to K$ are continuous, where $K \times K$ is endowed with the product topology.

(c) Show that the following are equivalent:

 (i) K is not Dedekind complete.

 (ii) K is disconnected, i.e., K can be expressed as the union of two disjoint, non-empty open subsets.

 (iii) K is totally disconnected, i.e., its only connected components are one-point sets.

Exercise 14
(4 points)
Let R be a real closed field and let $f(\mathrm{x}) = d_m \mathrm{x}^m + d_{m-1}\mathrm{x}^{m-1} + \ldots + d_0 \in R[\mathrm{x}]$ with $d_m \neq 0$. Show that the following statements are equivalent:

(i) $f \geq 0$ on R, i.e. $f(a) \geq 0$ for any $a \in R$.

(ii) $d_m > 0$ and all real roots of f, i.e. all roots of f in R, have even multiplicity.

(iii) $f = g^2 + h^2$ for some $g, h \in R[\mathrm{x}]$.

(iv) $f \in \sum R[\mathrm{x}]^2$.

Exercise 15
(4 points)
Let R be a real closed field and let $f(\mathrm{x}) = \mathrm{x}^m + d_{m-1}\mathrm{x}^{m-1} + \ldots + d_0$ be a monic polynomial over R. Suppose that all roots $a_1, \ldots, a_m$ of f are real. Show that

$$a_i \geq 0 \text{ for all } i \in \{1, \ldots, m\} \iff (-1)^{m-i} d_i \geq 0 \text{ for all } i \in \{0, \ldots, m-1\}.$$

Exercise 16
(4 points)

(a) Construct a countable field K and two orderings $\leq$ and $\leq'$ on K such that $(K, \leq)$ is Archimedean and $(K, \leq')$ is non-Archimedean.

(b) Let R be a real closed field and K a subfield of R. Show that

$$K^{\mathrm{ralg}} = \{\alpha \in R \mid \alpha \text{ is algebraic over } K\},$$

the relative algebraic closure of K in R, is real closed. Give an example of a real closed field R and a proper subfield $K \subsetneq R$ such that $K^{\mathrm{ralg}} = R$.

(c) Construct a countable *Archimedean* real closed field and a countable *non-Archimedean* real closed field.

Please hand in your solutions by ***Thursday, 24 November 2022, 10:00h*** *in the* ***postbox 14*** *or per e-mail to your tutor.*