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Real Algebraic Geometry I

Exercise Sheet 5
Counting roots

Exercise 17
(4 points)

(a) How many distinct orderings on $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ extend the ordering on $\mathbb{Q}$?

(b) How many distinct orderings on $\mathbb{Q}(\pi)$ extend the ordering on $\mathbb{Q}$?

Justify your answers!

Exercise 18
(4 points)
Let $(K,<)$ be an ordered field such that for any $f(\mathrm{x}) \in K[\mathrm{x}]$ the intermediate value property holds, i.e. for any $a, b \in K$ with $a < b$

$$f(a) < 0 < f(b) \Longrightarrow \exists c \in \,]a,b[\colon f(c) = 0.$$

Show that K is real closed.

Exercise 19
(4 points)
Let R be a real closed field, let $f \in R[\mathrm{x}]$ be a non-constant polynomial. Let $a, b \in R$ be such that $a < b$ and $f(a)f(b) \neq 0$ and let $c \in]a,b[$ be such that $f(c) = 0$. Suppose $\gcd(f, f') = 1$. Show that there exists $\delta \in R$ such that for all $x \in R$ we have

$$|x - c| < \delta \Rightarrow \operatorname{sign}(f(x)f'(x)) = \operatorname{sign}(x - c) = \begin{cases} -1 & \text{if } x < c \\ 0 & \text{if } x = c \\ 1 & \text{if } x > c. \end{cases}$$

Exercise 20
(4 points)
Let R be a real closed field and let $f(\mathrm{x}) = \mathrm{x}^3 + 6\mathrm{x}^2 - 16 \in R[\mathrm{x}]$.

(a) Compute the Sturm sequence of f.

(b) Show that f has three distinct roots in $[-6, 2]$.

(c) Denote the roots of f by $\alpha_1 < \alpha_2 < \alpha_3$. Show that $\alpha_1 \in [-6, -5]$, $\alpha_2 \in [-3, -1]$ and $\alpha_3 \in [1, 2]$.

Bonus Exercise – optional
(4 points)

In lectures 4 and 5 it was shown that if a field K is real closed, then K^2 is a positive cone. In this exercise you will show that the converse does not hold.

Definition. An ordered field K in which every non-negative element is a square, i.e., K^2 is a positive cone, is called a *Euclidean field*.

Let $K_0 := \mathbb{Q}$. Let $k_0 \in K_0$ be such that $\sqrt{k_0} \in \mathbb{R} \setminus K_0$. Define $K_1 := \{a_0 + b_0\sqrt{k_0} \mid a_0, b_0 \in K_0\}$. Now let $k_1 \in K_1$ be such that $\sqrt{k_1} \in \mathbb{R} \setminus K_1$ and let $K_2 := \{a_1 + b_1\sqrt{k_1} \mid a_1, b_1 \in K_1\}$. Keep proceeding this way to find a tower of field extensions $\mathbb{Q} = K_0 \subset K_1 \subset \ldots \subset K_i \subset \ldots$.

Definition. A number $\alpha \in \mathbb{R}$ is called *constructible* if there exists $n \in \mathbb{N}$ and a sequence $\mathbb{Q} = K_0 \subset K_1 \subset \ldots \subset K_i \subset \ldots$ as above, such that $\alpha \in K_n$. The set $\mathcal{K} = \{\alpha \in \mathbb{R} \mid \alpha \text{ is constructible}\}$ forms a field.

(a) Show that $\mathcal{K}$ is a euclidean field.

(b) Show that, for every $\alpha \in \mathcal{K}$, there exists $r \in \mathbb{N}_0$ such tht $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^r$.

(c) Deduce that $\mathcal{K}$ is not real closed.

Hint: find a cubic polynomial that has no root in $\mathcal{K}$.

*Please hand in your solutions by **Thursday, 1 December 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*