Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 6
Extensions of orderings and semialgebraic sets

Exercise 21
(4 points)
Let (K, P) be an ordered field and let R be the real closure of (K, P). Recall that R can be equipped with the order topology and K can be considered as a topological subspace of R.

(a) Suppose that (K, P) is Archimedean. Show that K is dense in R, i.e. that R is the topological closure of K in R.

(b) Construct an ordered field which is not dense in its real closure.

(Hint: Take a suitable field Q and consider $Q(\mathrm{x})$ with a suitable ordering.)

Exercise 22
(4 points)

(a) Let (K, P) be and ordered field, R a real closure and let L/K be a finite algebraic extension. Let $\alpha \in L$ be a primitive element: $L = K(\alpha)$ and let $f = \mathrm{MinPol}(\alpha/K)$ be the minimal polynomial of α over K.

- Show that there exists a bijection between the sets

$$\{\psi \colon L \hookrightarrow R \mid \psi \text{ is an embedding and } \psi|_K = \mathrm{id}_K\} \qquad \longleftrightarrow \qquad \{\beta \in R \mid f(\beta) = 0\}.$$

- Use this to deduce Corollary 2.7 from Corollary 2.6 (lecture 8).

(b) Let $\alpha = i\sqrt[4]{2}$ and $L = \mathbb{Q}(\alpha)$.

- Show that $\mathrm{MinPol}(\alpha/\mathbb{Q}) = x^4 - 2$;
- Compute the number of extensions of the natural order $\leq$ on $\mathbb{Q}$ to L.

Exercise 23
(4 points)
Let R be a real closed field and let $S(\underline{T}, \underline{X})$ be the system

$$T_2X_1^2 + T_1X_2^2 + T_1T_2X_1 + 1 = 0,$$

where $\underline{T} = (T_1, T_2)$ and $\underline{X} = (X_1, X_2)$. Find systems of equalities and inequalities $S_1(\underline{T}), \dots, S_\ell(\underline{T})$ with coefficients in $\mathbb{Q}$ such that

$$\forall \underline{T} \in R^2 : \left[\left(\exists \underline{X} \in R^2 : S\left(\underline{T}, \underline{X}\right) \right) \Longleftrightarrow \bigvee_{i=1}^{\ell} S_i\left(\underline{T}\right) \right].$$

Exercise 24
(4 points)
Let R be a real closed field.

(a) Show that the semialgebraic sets in R are exactly the finite unions of points in R and open intervals with endpoints in $R \cup \{\infty, -\infty\}$.

(b) Let $m \in \mathbb{N}$ and let A be a semialgebraic subset of R^m. Show that for some $n \in \mathbb{N}$, there is an algebraic set $B \subseteq R^{m+n}$ such that $\pi(B) = A$, where $\pi : R^{m+n} \to R^m$ is the projection map introduced in Lecture 11.

(Hint: Find a polynomial $f \in R[\underline{t}, \underline{x}]$, where $\underline{t} = (t_1, \dots, t_m)$ and $\underline{x} = (x_1, \dots, x_n)$, such that $A = \{\underline{t} \in R^m \mid \exists \underline{x} \in R^n : f(\underline{t}, \underline{x}) = 0\}$.)

*Please hand in your solutions by **Thursday, 8 December 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*