Fachbereich Mathematik und Statistik
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WS 2022 / 2023

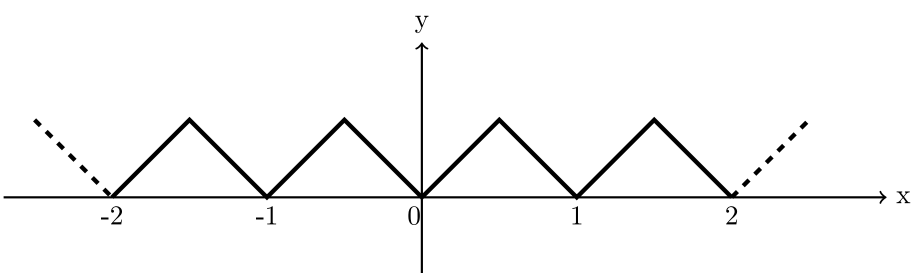
Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 7
Tarski–Seidenberg principle

Exercise 25
(4 points)
Let $n \in \mathbb{N}$ and $0 \neq f \in \mathbb{R}[x_1, \ldots, x_n]$. Show that $\mathbb{R}^n \setminus Z(f)$ is dense in $\mathbb{R}^n$. Is this still true if we replace $\mathbb{R}$ by any real closed field R?

Exercise 26
(4 points)
Let $Z \subseteq \mathbb{R}^2$ be the following zigzag curve in $\mathbb{R}^2$.



(a) Show that Z is not semialgebraic.

(b) Let A be a compact semialgebraic subset of $\mathbb{R}^2$. Show that $A \cap Z$ is semialgebraic.

Exercise 27
(4 points)
Let $\Phi(x)$ be the following first order formula in the language of real closed fields with one free variable x:

$$\neg \forall y \ \left(\exists z \left[(\neg\ x_3 = 0) \wedge y^2 + xy - z^2 - 1 = 0 \right] \vee -y^2 - xy + 1 > 0 \right).$$

Find a quantifier free first order formula in the language of real closed fields $\Psi(x)$ such that $\Phi(x) \sim \Psi(x)$.

Exercise 28
(4 points)
Let R be a real closed field. Decide which of the following sets A are **definable** in R (see Definition 2.2 of Lecture 12). For the definable sets, decide whether they can be defined without parameters, i.e. whether there is a first order formula $\Phi(\underline{X})$ with free variables $\underline{X} = (X_1, \dots, X_n)$ such that $A = \{\underline{r} \in R^n \mid \Phi(\underline{r}) \text{ is true in } R\}$. Justify your answers!

(a) $R = \mathbb{R}$, $f(x) = \exp(-x^2) - \frac{1}{\mathrm{e}^2}$ and $A = \{x \in \mathbb{R} \mid f(x) < 0\}$.

(b) $R = \mathbb{R}$ and $A = \{\sqrt{\pi}\}$.

(c) R the real closure of $\mathbb{R}(\mathrm{x})$ with the order induced by $\mathrm{x} > \mathbb{N}$ and $A = \{a \in R \mid a < n \text{ for some } n \in \mathbb{N}\}$.

(d) $R = \mathbb{R}$, $\pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x, y)$ and $A = \pi(B)$ for $B = \{(2\sin\theta, 2\cos\theta, \theta) \mid \theta \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

*Please hand in your solutions by **Thursday, 15 December 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*