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Real Algebraic Geometry I

Exercise Sheet 8
Semialgebraic sets II

Exercise 29
(4 points)

Let $n, s \in \mathbb{N}$ and let $f_i(\underline{T}, X)$ for $i = 1, \ldots, s$ be a sequence of polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$. For each of the following statements A_k, show that there exists a boolean combination $B_k(\underline{T}) = S_{k,1}(\underline{T}) \vee \ldots \vee S_{k,p}(\underline{T})$ of polynomial equations and inequalities in the variables $\underline{T}$ with coefficients in $\mathbb{Z}$, such that for any real closed field R and any $\underline{t} \in R^n$ we have that $A_k(\underline{t})$ holds true if and only if $B_k(\underline{t})$ holds true in R.

(a) $A_1(\underline{t})$: Exactly one of the polynomials $f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)$ has a zero in R.

(b) $A_2(\underline{t})$: Each of the polynomials $f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)$ has the same number of distinct zeros in R.

(c) $A_3(\underline{t})$: The polynomials $f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)$ have pairwise distinct zeros, i.e. no two of these polynomials have a common zero.

(d) $A_4(\underline{t})$: For any $x \in R$,

$$|\{i \in \{1, \ldots, s\} \mid f_i(\underline{t}, x) > 0\}| = |\{i \in \{1, \ldots, s\} \mid f_i(\underline{t}, x) < 0\}|,$$

i.e. the number of polynomials amongst $f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)$ which are positive in x is equal to the number of those which are negative in x.

Exercise 30
(4 points)

Let R be a real closed field.

(a) Let $n \in \mathbb{N}$ and let $A \subseteq R^n$ be a semialgebraic set. Recall that $||\underline{x}|| = \sqrt{x_1^2 + \ldots + x_n^2}$. Show that the following sets are semialgebraic.

- the closure of A: $\mathrm{cl}(A) = \{\underline{x} \in R^n \mid \forall t \in R\ \exists \underline{y} \in A\ (||\underline{y} - \underline{x}||^2 < t^2 \vee t = 0)\}$,
- the interior of A: $\mathrm{int}(A) = \{\underline{x} \in A \mid \exists t \in R\ \forall \underline{y} \in R^n\ (||\underline{y} - \underline{x}||^2 < t^2 \Rightarrow \underline{y} \in A)\}$,

- the boundary of A in R: $\partial A = \{\underline{x} \in R^n \mid \forall t \in R\ \exists \underline{y} \in A\ (||\underline{y} - \underline{x}||^2 < t^2) \wedge \exists \underline{z} \in R^n \setminus A\ (||\underline{z} - \underline{x}||^2 < t^2)\}$.

(b) Describe the closure $\mathrm{cl}(A)$ of the semialgebraic set

$$A = \left\{(x, y) \in R^2 \mid x^3 - x^2 - y^2 > 0\right\}.$$

Exercise 31
(4 points)
Let R be a real closed field. Let $A \subseteq R^n$, $B \subseteq R^m$ be semialgebraic sets for some $n, m \in \mathbb{N}$.

(a) Show that any polynomial map $f : A \to R$, i.e. any map of the form $f = p|_A$ for some $p \in R[X_1, \ldots X_n]$, is semialgebraic.

(b) Show that any regular rational map $f : A \to B$, i.e. a map of the form

$$f = \left(\frac{g_1}{h_1}, \ldots, \frac{g_m}{h_m}\right)$$

with $g_i, h_i \in R[X_1, \ldots X_n]$ and $h_i(\underline{a}) \neq 0$ for any $\underline{a} \in A$, is semialgebraic.

(c) Let $f, g : A \to R$ be semialgebraic maps. Show that the maps $\max(f, g) : x \mapsto \max(f(x), g(x))$, $\min(f, g) : x \mapsto \min(f(x), g(x))$ and $|f|$ are semialgebraic.

(d) Let $f : A \to R$ be a semialgebraic map with $f \geq 0$. Show that $\sqrt{f}$ is semialgebraic.

Exercise 32
(4 points)
Let R be a real closed field, let A, B, C be semialgebraic sets and let $f : A \to B$ and $g : B \to C$ be semialgebraic maps.

(a) Show that $g \circ f$ is semialgebraic.

(b) Show that for any semialgebraic subsets $S \subseteq A$ and $T \subseteq B$ also $f(S)$ and $f^{-1}(T)$ are semialgebraic.

(c) Let $\mathcal{S}(A) := \{f : A \to R \mid f \text{ is semialgebraic}\}$. Show that $\mathcal{S}(A)$ endowed with pointwise addition and multiplication is a commutative ring with an identity.

*Please hand in your solutions by **Thursday, 22 December 2022, 10:00h** in the **postbox 14** or per e-mail to your tutor.*