Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 9
Commutative Algebra

Exercise 33
(4 points)
Let A be the ring (with respect of pointwise addition and multiplication) of continuous functions $f : [0,1] \to \mathbb{R}$. Find a preordering T and an ordering P of A such that the following conditions are satisfied:

(a) There are infinitely many distinct preorderings T_i of A such that $\sum A^2 \subsetneq T_i \subsetneq T$.

(b) There are infinitely many distinct preorderings S_i of A such that $T \subsetneq S_i \subsetneq P$.

Exercise 34
(6 points)
Let A be a commutative ring with 1 such that $\frac{1}{2} \in A$ and let M be a quadratic module in A.

(a) Show that $M \cap (-M)$ is an ideal of A.

(b) Let $a \in A$. Show that the following are equivalent:

  (i) $a \in \sqrt{M \cap (-M)}$.
  (ii) $a^{2m} \in M \cap (-M)$ for some $m \in \mathbb{N}$.
  (iii) $-a^{2m} \in M$ for some $m \in \mathbb{N}$.

(c) Let I be an ideal of A and $T = \sum A^2 + I$. Show that $\sqrt[R]{I} = \sqrt{T \cap (-T)}$.

(d) Let $s, t \in \mathbb{N}$, $g_1, \ldots, g_s, h_1, \ldots, h_t \in A$, $S = \{g_1, \ldots, g_s\}$ and $I = \langle h_1, \ldots, h_t \rangle$. Suppose that M_S is a preordering in A. Show that $M_S + I$ is the preordering in A generated by $S \cup \{\pm h_i \mid i \in \{1, \ldots, t\}\}$.

Exercise 35
(4 points)

(a) Let A be a commutative ring with 1. Show that any prime ideal of A is radical.

(b) Find a field K and an ideal $I \subseteq K[X_1, \ldots, X_n]$ (for some $n \in \mathbb{N}$) such that I is radical but not prime.

(c) Find a field K and an ideal $I \subseteq K[X_1, \ldots, X_n]$ (for some $n \in \mathbb{N}$) such that I is prime but not real.

Exercise 36
(4 points)
Let K be a field and $A \subseteq K^n$ for some $n \in \mathbb{N}$.

(a) Show that:

   (i) $\mathcal{I}(A)$ is an ideal of $K[\underline{X}]$.

   (ii) If A is an algebraic set, then $\mathcal{Z}(\mathcal{I}(A)) = A$.

   (iii) The map $V \mapsto \mathcal{I}(V)$ is an injection from the set of of algebraic subsets of K^n into the set of ideals of $K[\underline{X}]$.

(b) (i) Show that for any ideal $I \subseteq K[\underline{X}]$, the inclusion $I \subseteq \mathcal{I}(\mathcal{Z}(I))$ holds.

   (ii) Find some ideal $I \subseteq K[\underline{X}]$ such that $I \neq \mathcal{I}(\mathcal{Z}(I))$.

*Please hand in your solutions by **Thursday, 12 January 2023, 10:00h** in the **postbox 14** or per e-mail to your tutor.*

Happy Christmas brake!