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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 10
PSD- and SOS polynomials

Exercise 37 **(4 points)**

Let $f \in \mathbb{R}[\underline{X}]$ be an sos form.

(a) Show that every sos representation of f consists of homogeneous polynomials, i.e. for any $f_1, \ldots, f_s \in \mathbb{R}[\underline{X}]$,

$$f = f_1^2 + \ldots + f_s^2 \implies f_1, \ldots, f_s \text{ are homogeneous.}$$

(b) Let $n, d \in \mathbb{N}$ and suppose that $f \in \Sigma_{n,d}$. Show that there is some $s \leq \binom{n+d}{d}$ such that f can be written as the sum of s squares.

Exercise 38 **(4 points)**

Let R be a real closed field.

(a) Let $f(X, Y) = X^6 + X^4Y^2 + 3X^2Y^4 + 3Y^6$. Write f as the sum of two squares in $R[X, Y]$.

(b) Let $g(X, Y, Z, T) = 2X^2 + 2XY + 2Y^2 + 3Z^2 + 2ZT + 3T^2$. Write g as the sum of four squares in $R[X, Y, Z, T]$.

Exercise 39 **(4 points)**

Let $p(\underline{X}) \in \mathbb{R}[\underline{X}]$ be of degree m. Show the following:

1. p is psd if and only if p_h is psd.
2. p is sos if and only if p_h is sos.

Exercise 40 **(4 points)**

Let R be a real closed field and $n, m \in \mathbb{N}$. We denote by $\mathcal{P}_{n,m}(R)$ the set of psd forms with coefficients in R of degree m in n variables and by $\Sigma_{n,m}(R)$ the set of sos forms with coefficients in R of degree m in n variables. Show the following:

(a) For every $d \in \mathbb{N}$, $\mathcal{P}_{2,2d}(R) = \Sigma_{2,2d}(R)$.

(b) For every $n \in \mathbb{N}$, $\mathcal{P}_{n,2}(R) = \Sigma_{n,2}(R)$.

(c) $\mathcal{P}_{3,4}(R) = \Sigma_{3,4}(R)$.

(Hint: Recall that by the Tarski Transfer Principle, any first order formula in the language of real closed fields without free variables transfers from $\mathbb{R}$ to any real closed field. For part (c) you may use that Hilbert proved that any $f \in \mathcal{P}_{3,4}(\mathbb{R})$ can be expressed as the sum of three squares.)

*Please hand in your solutions by **Thursday, 19 January 2023, 10:00h** in the **postbox 14** or per e-mail to your tutor.*