Fachbereich Mathematik und Statistik
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WS 2022 / 2023

Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 11
PSD- and SOS polynomials II

Exercise 41 **(4 points)**
The aim of this exercise is to prove the Spectral Theorem for real closed fields. Let R be a real closed field. Let $n \in \mathbb{N}$ and $M_n(R)$ the set of all $(n \times n)$-matrices with coefficients in R. Show that for every symmetric matrix $A \in M_n(R)$, there is a matrix $S \in M_n(R)$ and a diagonal matrix $D \in M_n(R)$ such that

$$S^T S = I \text{ and } A = SDS^T.$$

Exercise 42 **(4 points)**
Let K be a real closed field and let $0 \not\equiv f \in K[x_1, \ldots, x_n]$ be irreducible. Show that if f changes sign on K^n (i.e. $\exists\, x, y \in K^n$ s.t. $f(x)f(y) < 0$) then $(f) = \mathcal{I}(\mathcal{Z}(f))$, where (f) is the principal ideal generated by f and $\mathcal{I}(\mathcal{Z}(f))$ is the ideal of vanishing polynomials on the zero set of f.

Exercise 43 **(4 points)**

(a) Show that $f(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \in \mathbb{R}[x, y]$ is not sos.
(Hint: Assume, for a contradiction, that f is sos and compare coefficients. Note that $f(x, 0) = f(0, y) = 1$.)

(b) Deduce that the Motzkin form $M(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathbb{R}[x, y, z]$ is not sos.

(c) Show that the ternary sextic

$$g(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$$

is psd but not sos.

Exercise 44 **(4 points)**
Show that for all $n \in \mathbb{N}$ and for all $\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n \in \mathbb{R}^{\geq 0} = [0, \infty[$,

$$\sum_{i=1}^{n} \alpha_i = 1 \implies \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i} \geq 0.$$

Please hand in your solutions by ***Thursday, 26 January 2023, 10:00h*** *in the* ***postbox 14*** *or per e-mail to your tutor.*