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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 12

Exercise 45 **(4 points)**
Show that the symmetric quaternary quartic $F \in \mathbb{R}[\underline{x}]$ given by

$$F(x_1, x_2, x_3, x_4) = \sum_{j=2}^{4} \sum_{i<j} x_i^2 x_j^2 + \sum_{k=2}^{4} \sum_{j<k} \sum_{k \neq i \neq j} x_i^2 x_j x_k - 2x_1 x_2 x_3 x_4$$

is psd but not sos.
(Hint: Recall Proposition 4.2 of Lecture 23.)

Exercise 46 **(4 points)**
Consider the following version of Robinson's lemma:

Lemma A polynomial $P(x, y, z)$ of degree at most 2 which vanishes at seven of the eight points $(x, y, z) \in \{0, 1\}^3$ must also vanish at the eighth point.

Show that the quaternary quartic:

$$P(x, y, z, w) := x^2(x - w)^2 + [y(y - w) - z(z - w)]^2 + 2yz(x + y - w)(x + z - w)$$

is psd but not sos.

Exercise 47 **(4 points)**
Let A be a commutative ring with 1 and let

$$\chi := \operatorname{Hom}(A, \mathbb{R}) = \{\alpha \colon A \to \mathbb{R} \mid \alpha \text{ is a ring homomorphism}\}.$$

Consider the map defined by

$$\begin{aligned} \chi &\to \operatorname{Sper} A \\ \alpha &\mapsto P_\alpha := \alpha^{-1}\left(\mathbb{R}^{\geq 0}\right), \end{aligned} \tag{1}$$

where $Sper(A) := \{P \ : \ P \text{ is an ordering of} A\}$.
Show that:

a) this map is well-defined, i.e. $P_\alpha \subseteq A$ is an ordering;

b) this map is injective, i.e. $\alpha \neq \beta \Rightarrow P_\alpha \neq P_\beta$;

c) $\mathrm{Supp}(P_\alpha) = \ker \alpha$

Exercise 48 **(4 points)**

Keep the notation of **Exercise 46**.
For every $a \in A$, define:

$$\begin{aligned} \hat{a}\colon \chi &\to \mathbb{R} \\ \alpha &\mapsto \hat{a}(\alpha) := \alpha(a) \end{aligned} \tag{2}$$

and

$$\mathcal{U}(\hat{a}) := \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}.$$

Show that:

d) the collection $\mathcal{B} := \{\, \mathcal{U}(\hat{a}) \mid a \in A\}$ is a sub-base for a topology τ on χ;

e) for every $a \in A$ the map $\hat{a}\colon \chi \to \mathbb{R}$ defined in (2) is continuous with the respect to the topology τ;

f) if τ_1 is another topology on χ such that $\hat{a}$ is continuous for every $a \in A$, then $\mathcal{U}(\hat{a}) \in \tau_1$ for every $a \in A$ (i.e. τ_1 has more open sets than τ).

[In other words, the topology τ is the *weakest topology* on χ for which the map $\hat{a}$ is continuous for every $a \in A$.]

g) if we endow $\mathrm{Sper}\, A$ with the spectral topology, then the topology induced by the map in (1) coincide with τ.

[Recall that a sub-basis of open sets for the spectral topology is given by the collection $\{u(a)\colon a \in A\}$ where $u(a) := \{P \in \mathrm{Sper}\, A\colon a \notin -P\}$.]

*Please hand in your solutions by **Thursday, 2 February 2023, 10:00h** in the **postbox 14** or per e-mail to your tutor.*