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Real Algebraic Geometry I

Sheet 13* – Solution hints

Exercise 49
Let A be a commutative ring with 1 containing $\mathbb{Q}$. Let T be a generating preprime and M a maximal proper T-module. Suppose M is Archimedean. Define the map

$$\begin{aligned} \alpha\colon A &\longrightarrow \mathbb{R} \\ a &\mapsto \inf\{r \in \mathbb{Q} : r - a \in M\}. \end{aligned}$$

Show that α is a ring homomorphism.

Solution hints: For the well definedness see Lecture 25.
To show $\alpha(0) = 0$ and $\alpha(1) = 1$, show that $\mathbb{Q}_+ \subseteq M$ and that $\mathbb{Q}_- \cap M = \{0\}$. Use then $0 = \inf\{r \in \mathbb{Q} | r \in M\} = \alpha(0)$.
To prove additivity: $\alpha(a+b) = \alpha(a) = \alpha(b)$ prove first
$U_a + U_b \subseteq U_{a+b}$
and use it to show $\alpha(a+b) \leq= \alpha(a) = \alpha(b)$.
For the opposite inequality define $I_a = \{r \in \mathbb{Q} \mid r - a \notin M\}$ then prove and use that
$I_a + I_b \subseteq I_{a+b}$.

Exercise 50
Let R be a real closed field and let $S(\underline{T}, \underline{X})$ be the system

$$T_2 X_1^2 + T_1 X_2^2 + T_1 T_2 X_1 - 1 = 0,$$

where $\underline{T} = (T_1, T_2)$ and $\underline{X} = (X_1, X_2)$. Find systems of equalities and inequalities $S_1(\underline{T}), \dots, S_\ell(\underline{T})$ with coefficients in $\mathbb{Q}$ such that

$$\forall \underline{T} \in R^2 : \left[\left(\exists \underline{X} \in R^2 : S\left(\underline{T}, \underline{X}\right) \right) \Longleftrightarrow \bigvee_{i=1}^{\ell} S_i\left(\underline{T}\right) \right].$$

Solution hints: Standard application of Tarski-Seidenberg. A solution is given by $\ell = 3$ and

$$S_1 = \begin{cases} T_2 = 0 \\ T_1 > 0 \end{cases} \quad ; \qquad S_2 = \begin{cases} T_2 \neq 0 \\ -4T_1T_2 = 0 \\ T_1^2T_2^2 + 4T_2 \geq 0 \end{cases} \quad ; \qquad S_3 = \begin{cases} T_2 \neq 0 \\ -4T_1T_2 \leq 0 \\ T_1^2T_2^2 + 4T_2 \geq 0 \end{cases}$$

Exercise 51
An ordered field $(K, \leq)$ is called **Euclidean** if any non-negative element has a square root in K, i.e. for any $x \in K$ with $x \geq 0$ there is some $y \in K$ such that $y^2 = x$. Construct a Euclidean ordered field which is *not* real closed.

Solution hints: This was actually already given as bonus exercise in Sheet 5 (with solution hints!)

Exercise 52

(a) Let T be the set of all elements in $\mathbb{R}$ which are transcendental over $\mathbb{Q}$. Show that there is a bijection between T and $\mathbb{R}$, i.e. that T and $\mathbb{R}$ have the same cardinality.

(b) Show that for any set A, there is a set P_A with greater cardinality than A, i.e. there is no surjection from A to P_A. Deduce that there are at least countably infinitely many distinct uncountable cardinalities.

Solution hints:

(a) First show that the set of algebraic numbers is countable. For this, show that $\mathbb{Q}[X]$ and $\mathbb{Q}$ have the same cardinality. Thus use the fact that every algebraic number is the root of a finite degree polynomial over $\mathbb{Q}$. Each such polynomial has finitely many solutions. So the set of algebraic numbers is the union over all the polynomials over $\mathbb{Q}$ of their sets of roots. Thus it is countable.

    Since $\mathbb{R}$ is uncountable and the algebraic numbers are countable, the transcendental numbers must now be uncountable.

(b) Take for $P_A = \mathcal{P}(A)$, the set of all subsets of A. Then there is an injective map $A \to P_A,\ x \mapsto \{x\}$. Assume there was a surjective map $f\colon A \to P_A$. Let $M = \{a \in A \mid a \notin f(a)\} \in P_A$. By surjectivity there must be an $x \in A$ such that $M = f(x)$. And by definition of M we get $x \in f(x) = M \iff x \notin f(x)$. Which is a contradiction, so there is no surjective map from A to P_A. Thus P_A has cardinality strictly larger than A.

    To show the other statement start with $\mathbb{R}$ which has uncountable cardinality and form

    $$\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathcal{P}(\mathbb{R})), \ldots$$

    and apply (a) to prove that each of these has cardinality strictly larger than the last.

Exercise 53
Let S and S' be the subsets of $\mathbb{R}[X]$ given by $S = \{1 - X, 1 + X\}$, $S' = \{1 - X^2\}$, and let $K = [-1, 1] \subseteq \mathbb{R}$.

(a) Show that $K = K_S = K_{S'}$.

(b) Show that T_S and $T_{S'}$ are saturated.

Solution hints:
For part (a) just write explicitly the sets and compare them.
For part (b) put them in natural description and apply Proposition 3.3 Lecture 25.

Exercise 54
Let $(K, \leq)$ be an ordered field.

(a) Define a relation $\sim$ on K by $a \sim b$ if and only if there is some $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$. Show that $\sim$ defines an equivalence relation on K.

(b) Let $G = \{[a] \mid a \in K \setminus \{0\}\}$, i.e. the set of equivalence classes of $K \setminus \{0\}$ under $\sim$. Let v be the natural valuation on K, i.e. the map $v\colon\ K \to G \cup \{\infty\}, a \mapsto [a]$, where ∞ stands for $[0]$. Show that G is a group under addition given by $[a] + [b] = [ab]$. Show that $(G \cup \{\infty\}, \leq)$ is a totally ordered set, where the order relation is given by

$$[a] \leq [b] :\Longleftrightarrow [a] = [b] \vee |b| < |a|.$$

(c) Set $0 = [1]$. Let $\theta_v = \{a \in K \mid v(a) \geq 0\}$ and let $\mathcal{I}_v = \{a \in K \mid v(a) > 0\}$. Show that θ_v is a ring and that $\mathcal{I}_v$ is a maximal ideal of θ_v.

Solution hints:

(a) Straightforward

(b) Remember to check that the operation is well defined, i.e., it only depends on the classes and not on the particular representative. Moreover, be careful on what the neutral element should be. As for the ordering, reflexivity and antisimmetry are apparent. Transitivity and totality follow directly from the same properties on the ordering on K.

(c) This gives away the neutral element in (b). The key property to prove here is that, for all $a, b \in K$ one has $v(a + b) \geq \min\{v(a), v(b)\}$. Then check that θ_v and $\mathcal{I}_v$ are additive subgroups of K and that θ_v is also closed under products, while $\mathcal{I}_v$ is closed under multiplication by elements of θ_v.