Fachbereich Mathematik und Statistik
Dr. Michele Serra
Prof. Dr. Salma Kuhlmann
SS 2023

Universität Konstanz

Introduction to Elliptic Curves

Exercise Sheet 5
Heights

Recall the definition of the *logarithmic height* on $\mathbb{Q}$:

$$\forall x = \frac{a}{b} \in \mathbb{Q}, \qquad h(x) = \log H(x)$$

where a/b is in reduced form and $H(x) = \max\{|a|, |b|\}$.
Let E be an elliptic curve over $\mathbb{Q}$. Then we defined the *naïve height* on E by

$$h\colon E(\mathbb{Q}) \to \mathbb{R}_{\geq 0}, \qquad h(P) = \begin{cases} h(x) & \text{if } P = (x, y) \neq O \\ 0 & \text{if } P = O \end{cases}$$

Exercise 17 **(6 points)**
Show that there exists a constant $c_2 \in \mathbb{R}_{>0}$ such taht, fora all $P, Q \in E(\mathbb{Q})$ we have

$$h(P+Q) + h(P-Q) \leq 2h(P) + 2h(Q) + c_2$$

Hints: Assume wlog that $E : Y^2 = X^3 + AX + B$ and call $x_i = \frac{a_i}{b_i}$, $i = 1, 2, 3, 4$ the X-coordinate of $P, Q, P+Q$ and $P-Q$, respectively. Let $H_i = H(x_i)$ and $h_i = h(x_i)$. Use the identities from the proof of the parallelogram law (for points) and the properties of the height H proved in class.

Exercise 18 **(4 points)**
Show that there exists a constant $c_3 \in \mathbb{R}_{>0}$ such that , for all $P \in E(\mathbb{Q})$ we have

$$4h(P) \leq h(2P) + c_3$$

Hint: If $P = (x, y)$ with $x = a/b$ and $2P = (x', y')$ write x' as a fraction of polynomials $F(a, b)/G(a, b)$ and apply the properties of the height H.

Exercise 19 **(2+2+2 points)**
The *canonical height* $\hat{h}$ on E is defined, for all $P \in E(\mathbb{Q})$, by

$$\hat{h}(P) = \lim_{n\to\infty} \frac{1}{4^n} h(2^n P).$$

Show that

(a) The map $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ is well defined, i.e., the sequence $(4^{-n}h(2^nP))_{n\in\mathbb{N}}$ converges, for all $P \in E(\mathbb{Q})$.

Hint: Use Exercise 18.

(b) There exists a constant $c_0 > 0$ such that, for all $P \in E(\mathbb{Q})$ we have $|h(P) - \hat{h}(P)| \leq c_0$.

(c) The parallelogram law. For all $P, Q \in E(\mathbb{Q})$ we have

$$\hat{h}(P+Q) + \hat{h}(P-Q) = 2\hat{h}(P) + 2\hat{h}(Q).$$

Please hand in your solutions by ***Wednesday, 28 June 2023, 13:30h*** *in the* ***"envelope-postbox" by F409*** *or per e-mail to your tutor.*